8. cvičení - řešení

Definiční obory ve vzorovém řešení nejsou uvedeny (jde o základní záležitost), nicméně jsou důležité a je třeba se jim věnovat. Jsou uvedeny ve výsledcích.

Připomeňte si vzorce pro derivace: https://www2.karlin.mff.cuni.cz/~kuncova/materialy/deriv-t.pdf.

Příklad 1 (a) $\int \sin^2 x + \cos^2 x \, dx = \int 1 \, dx \overset{c}{=} x$ (Pythagorova věta)
Jelikož $x' = 1$, tak $\int 1 \, dx \overset{c}{=} x$.

Příklad 1 (b) $\int 5x^7 + \frac{9}{x^2} \, dx \overset{\text{lin.}}{=} 5\int x^7 \, dx + 9\int x^{-2} \, dx \overset{c}{=} 5\frac{x^8}{8} + 9\frac{x^{-1}}{-1} = \frac{5}{8}x^8 - \frac{9}{x}$
Jelikož $(x^n)' = n \cdot x^{n-1}$ pro $n \neq -1$, tak $\int x^n \, dx = \frac{1}{n+1}x^{n+1}$ pro $n \neq -1$.

Příklad 1 (c) $\int \sqrt{x} + e^{2x} \, dx \overset{\text{lin.}}{=} \int x^{\frac{1}{2}} \, dx + \int e^{2x} \, dx \overset{c}{=} \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{1}{2}e^{2x} = \frac{2}{3}x^{\frac{3}{2}} + \frac{1}{2}e^{2x}$
Platí, že $\left(\frac{1}{2}e^{2x}\right)' = \frac{1}{2} \cdot 2 \cdot e^{2x} = e^{2x}$. Tedy skutečně $\int e^{2x} \, dx \overset{c}{=} \frac{1}{2}e^{2x}$.

Příklad 1 (d) $\int 2\sin 3x + e^{-x} + 4 \, dx \overset{\text{lin.}}{=} 2\int \sin 3x \, dx + \int e^{-x} \, dx + 4\int 1 \, dx \overset{c}{=} \frac{-2}{3}\cos 3x - e^{-x} + 4x$
Opět zřejmě $\left(\frac{-2}{3}\cos 3x\right)' = \frac{-2}{3} \cdot (-\sin 3x) \cdot 3 = 2\sin 3x$. Pak tedy platí, že $\int 2\sin 3x \, dx \overset{c}{=} \frac{-2}{3}\cos 3x$.

Příklad 1 (e) $\int x^{\frac{1}{5}} - \frac{2}{x^3} \, dx \overset{c}{=} \frac{x^{\frac{1}{5}+1}}{\frac{1}{5}+1} - 2\frac{x^{-3+1}}{-3+1} = \frac{5}{6}x^{\frac{6}{5}} + x^{-2}$

Příklad 1 (f) $\int (x+1)^2 + \cos\frac{x}{2} + \frac{7}{x} \, dx \overset{\text{lin.}}{=} \int x^2 + 2x + 1 \, dx + 2\sin\frac{x}{2} + 7\log|x| \overset{c}{=} \frac{1}{3}x^3 + x^2 + x + 2\sin\frac{x}{2} + 7\log|x|$

Příklad 1 (g) $\int (2-x)^4 - \frac{1}{1+x^2} \, dx = \int (x-2)^4 - \frac{1}{1+x^2} \, dx \overset{c}{=} \frac{(x-2)^5}{5} - \arctan x$

Příklad 1 (h)

$$\int \frac{x^3 + 4x + 1}{2\sqrt{x}} \, dx \overset{\text{lin.}}{=} \frac{1}{2}\int x^{3-\frac{1}{2}} \, dx + \frac{4}{2}\int x^{1-\frac{1}{2}} \, dx + \frac{1}{2}\int x^{-\frac{1}{2}} \, dx =$$
$$\overset{c}{=} \frac{1}{2}\frac{x^{\frac{5}{2}+1}}{\frac{5}{2}+1} + 2\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{1}{2}\frac{x^{\frac{-1}{2}+1}}{\frac{-1}{2}+1} = \frac{1}{2}\frac{x^{\frac{7}{2}}}{\frac{7}{2}} + 2\frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{1}{2}\frac{x^{\frac{1}{2}}}{\frac{1}{2}} =$$
$$= \frac{1}{7}x^{\frac{7}{2}} + \frac{4}{3}x^{\frac{3}{2}} + \sqrt{x}$$

Příklad 1 (i) $\int \frac{1}{\sqrt{1-x^2}} \, dx \overset{c}{=} \arcsin x$

Příklad 2 (a)

$$\int 2xe^{-x} \, dx \overset{\text{lin.}}{=} 2\int x \cdot e^{-x} \, dx \overset{\text{PP}}{=} \left|u = x, u' = 1, v' = e^{-x}, v = -e^{-x}\right| =$$
$$= 2(-xe^{-x}) - \int 1 \cdot (-e^{-x}) \, dx = -2xe^{-x} + \int e^{-x} \, dx \overset{c}{=} -2xe^{-x} - e^{-x}$$

Příklad 2 (b)

$$\int x\sin x \, \mathrm{d}x \overset{\text{PP}}{=} \left|u = x, u' = 1, v = -\cos x, v' = \sin x\right| = -x\cos x - \int 1\cdot(-\cos x) \, \mathrm{d}x =$$
$$\overset{\text{c}}{=} \sin x - x\cos x$$

Příklad 2 (c)

$$\int (3x^3+x^2+1)e^{3x} \, \mathrm{d}x \overset{\text{PP}}{=} \left|u = 3x^3+x^2+1, u' = 9x^2+2x, v = \frac{1}{3}e^{3x}, v' = e^{3x}\right| =$$
$$= (3x^3+x^2+1)e^{3x}\frac{1}{3} - \frac{1}{3}\int (9x^2+2x)e^{3x} \, \mathrm{d}x \overset{\text{PP}}{=} \left|u = 9x^2+2x, u' = 18x+2, v = \frac{1}{3}e^{3x}, v' = e^{3x}\right| =$$
$$= (3x^3+x^2+1)e^{3x}\frac{1}{3} - \frac{1}{3}\left(\frac{1}{3}(9x^2+2x)e^{3x} - \frac{1}{3}\int (18x+2)e^{3x} \, \mathrm{d}x\right) =$$
$$\overset{\text{PP}}{=} \left|u = 18x+2, u' = 18, v = \frac{1}{3}e^{3x}, v' = e^{3x}\right| =$$
$$= (3x^3+x^2+1)\frac{e^{3x}}{3} - \frac{e^{3x}}{9}(9x^2+2x) + \frac{1}{9}\left(\frac{1}{3}(18x+2)e^{3x} - \frac{18}{3}\int e^{3x} \, \mathrm{d}x\right) =$$
$$\overset{\text{c}}{=} (3x^3+x^2+1)\frac{e^{3x}}{3} - \frac{e^{3x}}{9}(9x^2+2x) + \frac{e^{3x}}{27}(18x+2) - \frac{2}{3}\cdot\frac{1}{3}e^{3x} =$$
$$= e^{3x}\left(x^3 + \frac{x^2}{3} + \frac{1}{3} - x^2 - \frac{2}{9}x + \frac{2}{3}x + \frac{2}{27} - \frac{2}{9}\right) = e^{3x}\left(x^3 - \frac{2}{3}x^2 + \frac{4}{9}x + \frac{5}{27}\right)$$

Příklad 2 (d)

$$\int e^x\cos x \, \mathrm{d}x \overset{\text{PP}}{=} \left|u = \cos x, u' = -\sin x, v = e^x, v' = e^x\right| = e^x\cos x + \int e^x \sin x \, \mathrm{d}x =$$
$$\overset{\text{PP}}{=} \left|u = \sin x, u' = \cos x, v = e^x, v' = e^x\right| = e^x\cos x + e^x\sin x - \int e^x\cos x \, \mathrm{d}x$$

Máme:

$$\int e^x\cos x \, \mathrm{d}x = e^x\cos x + e^x\sin x - \int e^x\cos x \, \mathrm{d}x \implies 2\int e^x\cos x \, \mathrm{d}x \overset{\text{c}}{=} e^x(\sin x + \cos x)$$

Platí tedy, že $\int e^x\cos x \, \mathrm{d}x \overset{\text{c}}{=} \frac{1}{2}e^x(\sin x + \cos x)$

Příklad 2 (e)

$$\int x\arctan x \, \mathrm{d}x \overset{\text{PP}}{=} \left|u = \arctan x, u' = \frac{1}{1+x^2}, v = \frac{x^2}{2}, v' = x\right| = \frac{x^2}{2}\arctan x - \frac{1}{2}\int \frac{x^2}{1+x^2} \, \mathrm{d}x =$$
$$= \frac{x^2}{2}\arctan x - \frac{1}{2}\int 1 - \frac{1}{1+x^2} \, \mathrm{d}x \overset{\text{c}}{=} \frac{x^2}{2}\arctan x - \frac{1}{2}(x - \arctan x)$$

Příklad 2 (f)

$$\int x^2 \log x \, \mathrm{d}x \overset{\text{PP}}{=} \left| u = \log x, u' = \frac{1}{x}, v = \frac{x^3}{3}, v' = x^2 \right| = \frac{x^3}{3} \log x - \frac{1}{3} \int x^2 \, \mathrm{d}x \overset{\text{c}}{=} \frac{x^3}{3} \log x - \frac{x^3}{9}$$

Příklad 2 (g)

$$\int \sqrt{x} \log^2 x \, \mathrm{d}x \overset{\text{PP}}{=} \left| u = \log^2 x, u' = 2 \log x \cdot \frac{1}{x}, v = \frac{2}{3} x^{\frac{3}{2}}, v' = \sqrt{x} \right| = \frac{2}{3} \sqrt{x^3} \log^2 x - \frac{4}{3} \int \log x \cdot x^{\frac{1}{2}} \, \mathrm{d}x =$$
$$\overset{\text{PP}}{=} \left| u = \log x, u' = \frac{1}{x}, v = \frac{2}{3} x^{\frac{3}{2}}, v' = \sqrt{x} \right| = \frac{2}{3} \sqrt{x^3} \log^2 x - \frac{4}{3} \left(\frac{2}{3} x^{\frac{3}{2}} \log x - \frac{2}{3} \int x^{\frac{1}{2}} \, \mathrm{d}x \right) =$$
$$\overset{\text{c}}{=} \frac{2}{3} \sqrt{x^3} \log^2 x - \frac{8}{9} x^{\frac{5}{2}} \log x + \frac{16}{27} x^{\frac{3}{2}}$$

Příklad 2 (h) $\int \log^2 x + \log x^2 \, \mathrm{d}x$

$$\int \log^2 x \, \mathrm{d}x \overset{\text{PP}}{=} \left| u = \log^2 x, u' = 2 \log x \frac{1}{x}, v = x, v' = 1 \right| = x \log^2 x - 2 \int \log x \, \mathrm{d}x =$$
$$\overset{\text{PP}}{=} \left| u = \log x, u' = \frac{1}{x}, v = x, v' = 1 \right| = x \log^2 x - 2x \log x + 2 \int 1 \, \mathrm{d}x \overset{\text{c}}{=} x \log^2 x - 2x \log x + 2x$$

$$\int \log x^2 \, \mathrm{d}x \overset{\text{PP}}{=} \left| u = \log x^2, u' = \frac{2x}{x^2} = \frac{2}{x}, v = x, v' = 1 \right| = x \log x^2 - 2 \int x \frac{1}{x} \, \mathrm{d}x \overset{\text{c}}{=} x \log x^2 - 2x$$

$$\int \log^2 x + \log x^2 \, \mathrm{d}x \overset{\text{lin.}}{=} \int \log^2 x \, \mathrm{d}x + \int \log x^2 \, \mathrm{d}x \overset{\text{c}}{=} x \log^2 x - 2x \log x + 2x + x \log x^2 - 2x =$$
$$= x \log^2 x - x \log x^2 + x \log x^2 = x \log^2 x$$

Příklad 2 (i)

$$\int \sin(\log 2x) \, \mathrm{d}x \overset{\text{PP}}{=} \left| u = \sin(\log 2x), u' = \cos(\log 2x) \frac{1}{x}, v = x, v' = 1 \right| =$$
$$= x \sin(\log 2x) - \int x \frac{1}{x} \cos(\log 2x) \, \mathrm{d}x \overset{\text{PP}}{=} \left| u = \cos(\log 2x), u' = -\sin(\log 2x) \frac{1}{x}, v = x, v' = 1 \right| =$$
$$= x \sin(\log 2x) - x \cos(\log 2x) - \int x \sin(\log 2x) \frac{1}{x} \, \mathrm{d}x = x \sin(\log 2x) - x \cos(\log 2x) - \int \sin(\log 2x) \, \mathrm{d}x$$

Máme tedy:

$$\int \sin(\log 2x) \, \mathrm{d}x = x \sin(\log 2x) - x \cos(\log 2x) - \int \sin(\log 2x) \, \mathrm{d}x \implies$$
$$\implies \int \sin(\log 2x) \overset{\text{c}}{=} \frac{1}{2} x \left(\sin(\log 2x) - \cos(\log 2x) \right)$$

Příklad 3(a)

$$\int (3x-2)^6 \ \mathrm{d}x = |y = 3x-2,\ \mathrm{d}y = 3\ \mathrm{d}x| = \int y^6 \frac{1}{3}\ \mathrm{d}y \overset{\mathrm{c}}{=} \frac{1}{3}\frac{y^7}{7} = \frac{(3x-2)^7}{21}$$

Poznámka: v substituci máme, že $\mathrm{d}y = 3\ \mathrm{d}x$. To vzniklo zderivováním rovnice $y = 3x-2$. Ze vztahu $\mathrm{d}y = 3\ \mathrm{d}x$ plyne: $\mathrm{d}x = \frac{1}{3}\ \mathrm{d}y$, což se objevuje v integrálu po provedení substituce.

Příklad 3(b)

$$\int \sin(2x+1)\ \mathrm{d}x = |y = 2x+1,\ \mathrm{d}y = 2\ \mathrm{d}x| = \int \sin y \cdot \frac{1}{2}\ \mathrm{d}x \overset{\mathrm{c}}{=} -\frac{1}{2}\cos y = -\frac{1}{2}\cos(2x+1)$$

Příklad 3(c)

$$\int \frac{1}{(x+1)^2+1}\ \mathrm{d}x = |y = x+1,\ \mathrm{d}y = \ \mathrm{d}x| = \int \frac{1}{y^2+1}\ \mathrm{d}y \overset{\mathrm{c}}{=} \arctan y = \arctan(x+1)$$

Příklad 3(d)

$$\int \frac{x}{(x^2+1)^2}\ \mathrm{d}x = \left|y = x^2+1,\ \mathrm{d}y = 2x\ \mathrm{d}x\right| = \int \frac{1}{2y^2}\ \mathrm{d}y \overset{\mathrm{c}}{=} \frac{1}{2}\cdot\frac{-1}{y} = -\frac{1}{2(x^2+1)}$$

Příklad 3(e)

Využijeme vztahu: $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$.

$$\int -\frac{1}{\sqrt{8-3x^2}}\ \mathrm{d}x \overset{\text{lin.}}{=} \frac{-1}{\sqrt{8}}\int \frac{1}{\sqrt{1-\frac{3}{8}x^2}}\ \mathrm{d}x = \left|y = \sqrt{\frac{3}{8}}x,\ \mathrm{d}y = \sqrt{\frac{3}{8}}\ \mathrm{d}x\right| \overset{\text{lin.}}{=} \frac{-1}{\sqrt{8}}\cdot\frac{\sqrt{8}}{\sqrt{3}}\int \frac{1}{\sqrt{1-y^2}}\ \mathrm{d}y =$$
$$\overset{\mathrm{c}}{=} -\frac{1}{\sqrt{3}}\arcsin y = -\frac{1}{\sqrt{3}}\arcsin\left(\sqrt{\frac{3}{8}}x\right)$$

Příklad 3(f)

$$\int 3xe^{-x^2}\ \mathrm{d}x = \left|y = -x^2,\ \mathrm{d}y = -2\ \mathrm{d}x\right| = \int \frac{-3}{2}e^y\ \mathrm{d}y \overset{\mathrm{c}}{=} \frac{-3}{2}e^y = -\frac{3}{2}e^{-x^2}$$

Příklad 3(g)

$$\int \frac{1}{(1+x^2)\arctan^3 x}\ \mathrm{d}x = \left|y = \arctan x,\ \mathrm{d}y = \frac{1}{1+x^2}\ \mathrm{d}x\right| = \int \frac{1}{y^3}\ \mathrm{d}y \overset{\mathrm{c}}{=} \frac{-1}{2y^2} = -\frac{1}{2\arctan^2 x}$$

Příklad 3(h)

$$\int \frac{1}{1+\sqrt{x-1}}\,\mathrm{d}x = \left| y=\sqrt{x-1},\ \mathrm{d}y = \frac{1}{2\sqrt{x-1}}\,\mathrm{d}x \right| = \int \frac{2y}{y+1}\,\mathrm{d}y \overset{\text{lin.}}{=} 2\int \frac{y+1-1}{y+1}\,\mathrm{d}y =$$
$$= 2\int 1-\frac{1}{y+1}\,\mathrm{d}y \overset{\text{lin.}}{=} 2y-2\int \frac{1}{y+1}\,\mathrm{d}y = |z=y+1,\ \mathrm{d}z=\ \mathrm{d}y| = 2y-2\int \frac{1}{z}\,\mathrm{d}z \overset{\text{c}}{=} 2y-2\log|z| =$$
$$= 2\sqrt{x-1}-2\log\left(1+\sqrt{x-1}\right)$$

Pozn.: absolutní hodnota není potřeba, neb $1+\sqrt{x-1}>0$, je-li výraz definován.

Příklad 3(i)
Uvědomme si: $\tan^5 x = \frac{\sin^5 x}{\cos^5 x}$ a $\sin^4 x = \left(\sin^2 x\right)^2 = \left(1-\cos^2 x\right)^2$.

$$\int \tan^5 x\,\mathrm{d}x = |y=\cos x,\ \mathrm{d}y = -\sin x\,\mathrm{d}x| = \int -\frac{\left(1-y^2\right)^2}{y^5}\,\mathrm{d}y = \int -\frac{1-2y^2+y^4}{y^5}\,\mathrm{d}y =$$
$$= \int -\frac{1}{y^5}+2\frac{1}{y^3}-\frac{1}{y}\,\mathrm{d}y \overset{\text{c}}{=} \frac{1}{4y^4}-\frac{1}{y^2}-\log|y| = \frac{1}{4\cos^4 x}-\frac{1}{\cos^2 x}-\log|\cos x|$$

Příklad 5(a)

$$\int \frac{1}{x^6}\sin\frac{1}{x^5}\,\mathrm{d}x = \left| y=\frac{1}{x^5},\ \mathrm{d}y = \frac{-5}{x^6} \right| \overset{\text{lin.}}{=} \frac{-1}{5}\int \sin y\,\mathrm{d}y \overset{\text{c}}{=} \frac{1}{5}\cos y = \frac{1}{5}\cos\frac{1}{x^5}$$

Příklad 5(b)

$$\int x^3\log 2x\,\mathrm{d}x \overset{\text{PP}}{=} \left| u=\log 2x, u'=\frac{1}{x}, v=\frac{x^4}{4}, v'=x^3 \right| = \frac{x^4}{4}\log 2x - \int \frac{x^4}{4}\frac{1}{x}\,\mathrm{d}x \overset{\text{c}}{=} \frac{x^4}{4}\log 2x - \frac{x^4}{16}$$

Příklad 5(c)

$$\int \left(1-\frac{1}{x^2}\right)\sqrt{x\sqrt{x}}\,\mathrm{d}x \overset{\text{lin.}}{=} \int \sqrt{x\sqrt{x}}\,\mathrm{d}x - \int \frac{\sqrt{x\sqrt{x}}}{x^2}\,\mathrm{d}x = \int x^{\frac{3}{4}}\,\mathrm{d}x - \int \frac{x^{\frac{3}{4}}}{x^2}\,\mathrm{d}x =$$
$$= \frac{4}{7}x^{\frac{7}{4}} - \int x^{-\frac{5}{4}}\,\mathrm{d}x \overset{\text{c}}{=} \frac{4}{7}x^{\frac{7}{4}} + 4x^{-\frac{1}{4}}$$

Příklad 5(d)
Nejdříve spočtěme: $\int 2xe^{-x^2}\,\mathrm{d}x = \left|y=-x^2,\ \mathrm{d}y=-2x\,\mathrm{d}x\right| = -\int e^y\,\mathrm{d}y = -e^y = -e^{-x^2}$

$$\int x^5e^{-x^2}\,\mathrm{d}x = \int \frac{-1}{2}x^4\cdot(-2x)e^{-x^2}\,\mathrm{d}x \overset{\text{PP}}{=} \left| u=\frac{-1}{2}x^4, u'=-2x^3, v=e^{-x^2}, v'=(-2x)e^{-x^2} \right| =$$
$$= \frac{-1}{2}x^4e^{-x^2} - \int e^{-x^2}(-2)x\cdot x^2\,\mathrm{d}x = \left|y=-x^2,\ \mathrm{d}y=-2x\,\mathrm{d}x\right| = -\frac{1}{2}x^4e^{-x^2} - \int e^y(-y)\,\mathrm{d}y =$$
$$\overset{\text{lin.}}{=} -\frac{1}{2}x^4e^{-x^2} + \int ye^y\,\mathrm{d}y \overset{\text{PP}}{=} \left|u=y, u'=1, v=e^y, v'=e^y\right| = -\frac{1}{2}x^4e^{-x^2} + ye^y - \int e^y\,\mathrm{d}y =$$
$$\overset{\text{c}}{=} -\frac{1}{2}x^4e^{-x^2} + ye^y - e^y = -\frac{1}{2}x^4e^{-x^2} + (-x^2)e^{-x^2} - e^{-x^2}$$